ECE 604, Lecture 17

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Contents

1 Generalized Transverse Resonance Condition **2**

2 Dielectric Waveguide **3**
2.1 TE Case 4
2.2 TM Case 9

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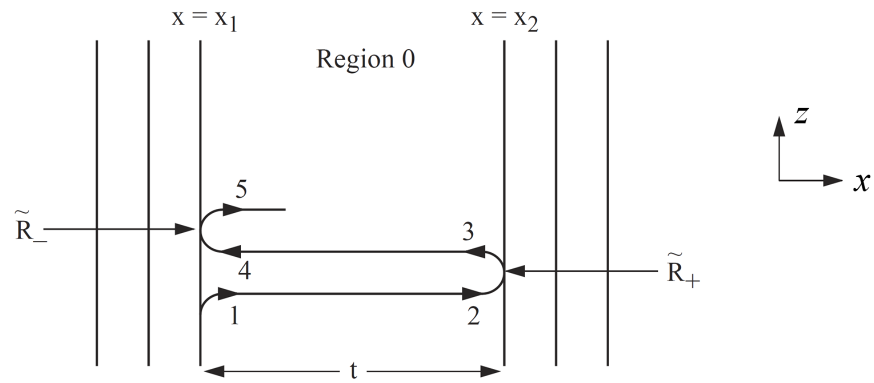

Figure 1: The transverse resonance condition for a layered medium. The phase of the wave at position 5 should be equal to the transverse phase at position 1.

1 Generalized Transverse Resonance Condition

The guidance conditions, the transverse resonance condition given previously, can also be derived for the more general case. The generalized transverse resonance condition is a powerful condition that can be used to derive the guidance condition of a mode in a layered medium.

To derive this condition, we first have to realize that a guided mode in a waveguide is due to the coherent or constructive interference of the waves. This implies that if a plane wave starts at position 1 (see Figure 1)[1] and is multiply reflected as shown, it will regain its original phase in the x direction at position 5. Since this mode progresses in the z direction. Moreover, waves at 1 and 5 will gain the same phase in the z direction. But, for it to coherently interfere in the x direction, the transverse phase at 5 must be the same as 1.

Assuming that the wave starts with amplitude 1 at position 1, it will gain a transverse phase of $e^{-j\beta_{0x}t}$ when it reaches position 2. Upon reflection at $x = x_2$, at position 3, the wave becomes $\tilde{R}_+ e^{-j\beta_{0x}t}$ where $\tilde{R}_+$ is the generalized reflection coefficient at the right interface of Region 0. Finally, at position 5, it becomes $\tilde{R}_- \tilde{R}_+ e^{-2j\beta_{0x}t}$ where $\tilde{R}_-$ is the generalized reflection coefficient at the left interface of Region 0. For constructive interference to occur or for the mode to exist, we require that

$$\tilde{R}_- \tilde{R}_+ e^{-2j\beta_{0x}t} = 1 \tag{1.1}$$

The above is the generalized transverse resonance condition for the guidance condition for a plane wave mode traveling in a layered medium.

In (1.1), a metallic wall has a reflection coefficient of 1 for a TM wave, hence

[1] The waveguide convention is to assume the direction of propagation to be z. Since we are analyzing a guided mode in a layered medium, z axis is as shown in this figure.

if $\tilde{R}_+$ is 1, Equation (1.1) becomes

$$1 - \tilde{R}_- e^{2-j\beta_{0x}t} = 0. \tag{1.2}$$

On the other hand, in (1.1), a metallic wall has a reflection coefficient of -1, for TE wave, and Equation (1.1) becomes

$$1 + \tilde{R}_- e^{2-j\beta_{0x}t} = 0. \tag{1.3}$$

2 Dielectric Waveguide

The most important dielectric waveguide of the modern world is the optical fiber, whose invention was credited to Charles Kao. He was awarded the Nobel prize in 2009. However, the analysis of the optical fiber requires analysis in cylindrical coordinates and the use of special functions such as Bessel functions. In order to capture the essence of dielectric waveguides, one can study the slab dielectric waveguide, which shares many salient features with the optical fiber. This waveguide is also used as thin-film optical waveguides (see Figure 2). We start with analyzing the TE modes in this waveguide.

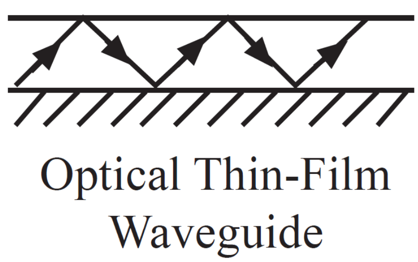

Figure 2:

2.1 TE Case

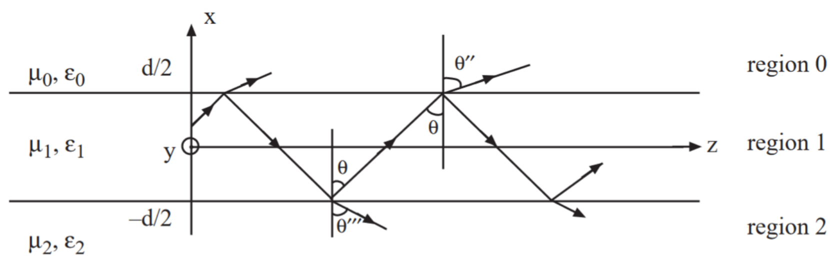

Figure 3:

We shall look at the application of the transverse resonance condition to a TE wave guided in a dielectric waveguide. Again, we assume the direction of propagation of the guided mode to be in the z direciton in accordance to convention. Specializing the above equation to the dielectric waveguide shown in Figure 3, we have the guidance condition as

$$1 = R_{10}R_{12}e^{-2j\beta_{1x}d} \tag{2.1}$$

where d is the thickness of the dielectric slab. Guidance of a mode is due to total internal reflection, and hence, we expect Region 1 to be optically more dense (in terms of optical refractive indices) than region 0 and 2.

To simplify the analysis further, we assume Region 2 to be the same as Region 0. The new guidance condition is then

$$1 = R_{10}^2 e^{-2j\beta_{1x}d} \tag{2.2}$$

Also, we assume that $\varepsilon_1 > \varepsilon_0$ so that total internal reflection occurs at both interfaces as the wave bounces around so that $\beta_{0x} = -j\alpha_{0x}$. Therefore, for TE polarization, the single-interface reflection coefficient is

$$R_{10} = \frac{\mu_0\beta_{1x} - \mu_1\beta_{0x}}{\mu_0\beta_{1x} + \mu_1\beta_{0x}} = \frac{\mu_0\beta_{1x} + j\mu_1\alpha_{0x}}{\mu_0\beta_{1x} - j\mu_1\alpha_{0x}} = e^{j\theta_{TE}} \tag{2.3}$$

where θ_{TE} is the Goos-Hanschen shift for total internal reflection. It is given by

$$\theta_{TE} = 2\tan^{-1}\left(\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}}\right) \tag{2.4}$$

The guidance condition for constructive interference according to (2.1) is such that

$$2\theta_{TE} - 2\beta_{1x}d = 2n\pi \tag{2.5}$$

From the above, dividing it by four, and taking its tangent, we get

$$\tan\left(\frac{\theta_{TE}}{2}\right) = \tan\left(\frac{n\pi}{2} + \frac{\beta_{1x}d}{2}\right) \tag{2.6}$$

or

$$\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}} = \tan\left(\frac{n\pi}{2} + \frac{\beta_{1x}d}{2}\right) \tag{2.7}$$

The above gives rise to

$$\mu_1\alpha_{0x} = \mu_0\beta_{1x}\tan\left(\frac{\beta_{1x}d}{2}\right), \qquad n \text{ even} \tag{2.8}$$

$$-\mu_1\alpha_{0x} = \mu_0\beta_{1x}\cot\left(\frac{\beta_{1x}d}{2}\right), \qquad n \text{ odd} \tag{2.9}$$

It can be shown that when n is even, the mode profile is even, whereas when n is odd, the mode profile is odd. The above can also be rewritten as

$$\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\tan\left(\frac{\beta_{1x}d}{2}\right) = \frac{\alpha_{0x}d}{2}, \qquad \text{even modes} \tag{2.10}$$

$$-\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\cot\left(\frac{\beta_{1x}d}{2}\right) = \frac{\alpha_{0x}d}{2}, \qquad \text{odd modes} \tag{2.11}$$

Using the fact that $-\alpha_{0x}^2 = \beta_0^2 - \beta_z^2$, and that $\beta_{1x}^2 = \beta_1^2 - \beta_z^2$, eliminating β_z from these two equations, one can show that

$$\alpha_{0x} = [\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0) - \beta_{1x}^2]^{\frac{1}{2}} \tag{2.12}$$

and (2.10) and (2.11) become

$$\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\tan\left(\frac{\beta_{1x}d}{2}\right) = \frac{\alpha_{0x}d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\frac{\beta_{1x}d}{2}\right)^2}, \qquad \text{even modes} \tag{2.13}$$

$$-\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\cot\left(\frac{\beta_{1x}d}{2}\right) = \frac{\alpha_{0x}d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\frac{\beta_{1x}d}{2}\right)^2}, \qquad \text{odd modes} \tag{2.14}$$

We can solve the above graphically by plotting

$$y_1 = \frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\tan\left(\frac{\beta_{1x}d}{2}\right) \qquad \text{even modes} \tag{2.15}$$

$$y_2 = -\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\cot\left(\beta_{1x}\frac{d}{2}\right) \qquad \text{odd modes} \tag{2.16}$$

$$y_3 = \left[\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\frac{\beta_{1x}d}{2}\right)^2\right]^{\frac{1}{2}} = \frac{\alpha_{0x}d}{2} \tag{2.17}$$

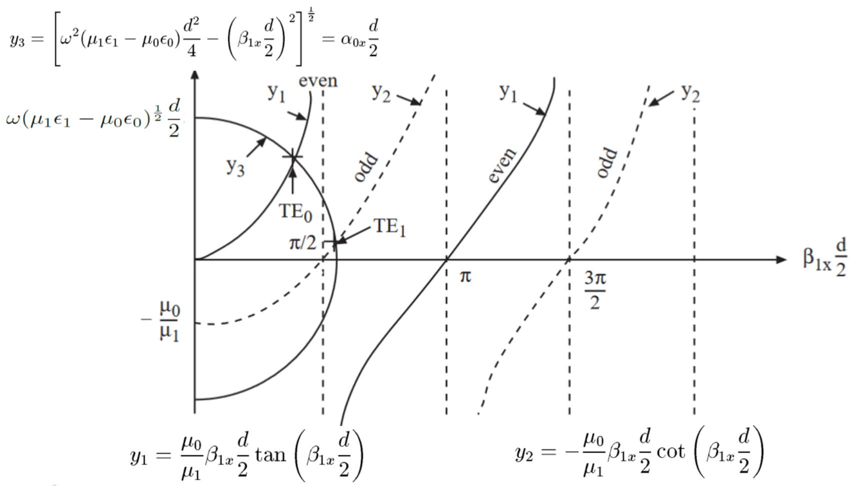

Figure 4:

In the above, y_3 is the equation of a circle; the radius of the circle is given by

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2}. \tag{2.18}$$

The solutions to (2.13) and (2.14) are given by the intersections of y_3 with y_1 and y_2. We note from (2.1) that the radius of the circle can be increased in three ways: (i) by increasing the frequency, (ii) by increasing the contrast $\frac{\mu_1\epsilon_1}{\mu_0\epsilon_0}$, and (iii) by increasing the thickness d of the slab.[2] The mode profiles of the first two modes are shown in Figure 5.

[2]These features are also shared by the optical fiber.

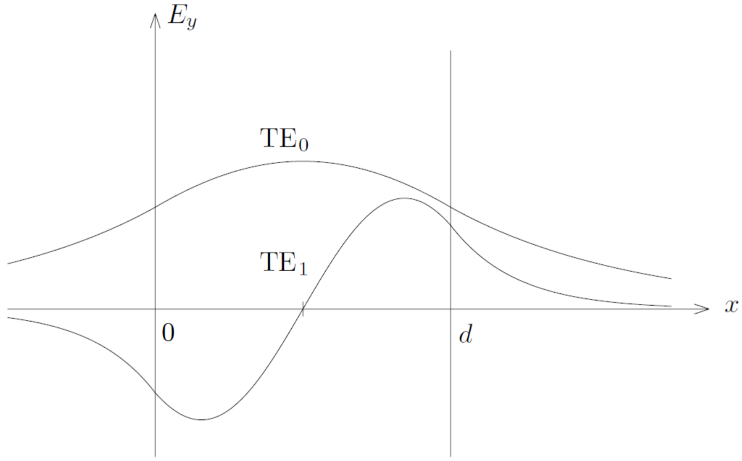

Figure 4.2.16 Field amplitudes for TE_0 and TE_1 modes.

Figure 5: Courtesy of J.A. Kong.

When $\beta_{0x} = -j\alpha_{0x}$, the reflection coefficient for total internal reflection is

$$R_{10}^{TE} = \frac{\mu_0\beta_{1x} + j\mu_1\alpha_{0x}}{\mu_0\beta_{1x} - j\mu_1\alpha_{0x}} = \exp\left[+2j\tan^{-1}\left(\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}}\right)\right] \tag{2.19}$$

and $\left|R_{10}^{TE}\right| = 1$. Hence, the wave is guided by total internal reflections.

Cut-off occurs when the total internal reflection ceases to occur, i.e. when the frequency decreases such that $\alpha_{0x} = 0$.

From Figure 4, we see that $\alpha_{0x} = 0$ when

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2} = \frac{m\pi}{2}, \qquad m = 0, 1, 2, 3, \ldots \tag{2.20}$$

or

$$\omega_{mc} = \frac{m\pi}{d(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}}, \qquad m = 0, 1, 2, 3, \ldots \tag{2.21}$$

The mode that corresponds to the m-th cut-off frequency above is labeled the TE_m mode. Thus TE_0 mode is the mode that has no cut-off or propagates at all frequencies. This is shown in Figure 6 where the TE mode profiles are similar since they are dual to each other. The boundary conditions at the dielectric interface is that the field and its normal derivative have to be continuous. The TE_0 or TM_0 mode can satisfy this boundary condition at all frequencies, but not the TE_1 or TM_1 mode. At the cut-off frequency, the field outside the slab has to become flat implying the $\alpha_{0x} = 0$ implying no guidance.

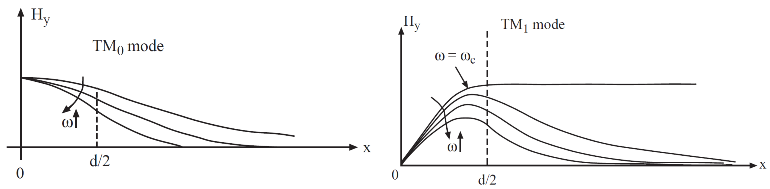

Figure 6: The TE modes are dual to the TM modes and have similar mode profiles.

At cut-off, $\alpha_{0x} = 0$, and from the dispersion relation that $\alpha_{0x}^2 = \beta_z^2 - \beta_0^2$,

$$\beta_z = \omega\sqrt{\mu_0 \epsilon_0},$$

for all the modes. Hence, both the group and the phase velocities are that of the outer region. This is because when $\alpha_{0x} = 0$, the wave is not evanescent outside, and most of the energy of the mode is carried by the exterior field.

When $\omega \to \infty$, the radius of the circle in the plot of y_3 becomes increasingly larger. As seen from Figure 4, the solution for $\beta_{1x} \to \frac{n\pi}{d}$ for all the modes. From the dispersion relation for Region 1,

$$\beta_z = \sqrt{\omega^2 \mu_1 \epsilon_1 - \beta_{1x}^2} \approx \omega\sqrt{\mu_1 \epsilon_1}, \qquad \omega \to \infty \tag{2.22}$$

Hence the group and phase velocities approach that of the dielectric slab. This is because when $\omega \to \infty$, $\alpha_{0x} \to \infty$, implying that the fields are trapped or confined in the slab and propagating within it. Because of this, the dispersion diagram of the different modes appear as shown in Figure 7. In this figure, k_{c1}, k_{c2}, and k_{c3} are the cut-off wave number or frequency of the first three modes. Close to cut-off, the field is traveling mostly outside the waveguide, and $k_z \approx \omega\sqrt{\mu_0 \varepsilon_0}$, and both the phase and group velocities approach that of the outer medium as shown in the figure. When the frequency increases, the mode is tightly confined in the dielectric slab, and $k_z \approx \omega\sqrt{\mu_1 \varepsilon_1}$. Both the phase and group velocities approach that of Region 1 as shown.

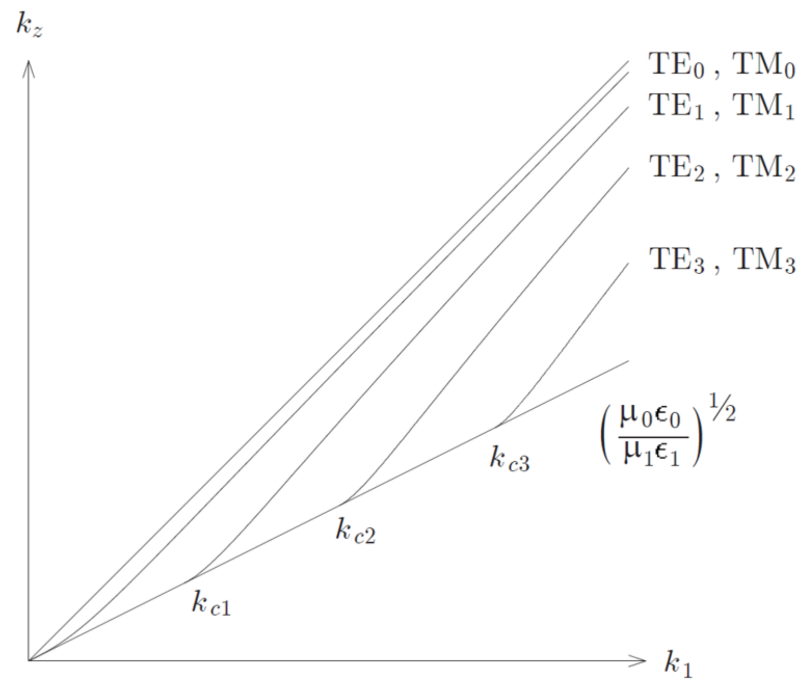

Figure 7: k_z versus k_1 plot for dielectric slab waveguide (Courtesy of J.A. Kong).

2.2 TM Case

For the TM case, a similar guidance condition analogous to (2.1) can be derived but with the understanding that the reflection coefficients in (2.1) are now TM reflection coefficients. Similar derivations show that the above guidance condition, for $\epsilon_2 = \epsilon_0$, $\mu_2 = \mu_0$, reduces to

$$\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\tan\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2}, \qquad \text{even modes} \tag{2.23}$$

$$-\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\cot\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2}, \qquad \text{odd modes} \tag{2.24}$$

Note that for equation (2.1), when we have two parallel metallic plates, $R^{TM} = 1$, and $R^{TE} = -1$, and the guidance condition becomes

$$1 = e^{-2j\beta_{1x}d} \quad \Rightarrow \beta_{1x} = \frac{m\pi}{d}, \qquad m = 0, 1, 2, \ldots, \tag{2.25}$$